

Penalizing null recurrent diffusions

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Abstract: We present some limit theorems for the normalized laws (with respect to functionals involving last passage times at a given level a up to time t) of a large class of null recurrent diffusions. Our results rely on hypotheses on the Lévy measure of the diffusion inverse local time at 0. As a special case, we recover some of the penalization results obtained by Najnudel, Roynette and Yor in the (reflected) Brownian setting.

Keywords: Penalization, null recurrent diffusions, last passage times, inverse local time.

1 Introduction

1.1 A few notation

We consider a linear regular null recurrent diffusion $(X_t, t \geq 0)$ taking values in \mathbb{R}^+ , with 0 an instantaneously reflecting boundary and $+\infty$ a natural boundary. Let \mathbb{P}_x and \mathbb{E}_x denote, respectively, the probability measure and the expectation associated with X when started from $x \geq 0$. We assume that X is defined on the canonical space $\Omega := \mathcal{C}(\mathbb{R}_+ \rightarrow \mathbb{R}_+)$ and we denote by $(\mathcal{F}_t, t \geq 0)$ its natural filtration, with $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t$.

We denote by s its scale function, with the normalization $s(0) = 0$, and by $m(dx)$ its speed measure, which is assumed to have no atoms. It is known that $(X_t, t \geq 0)$ admits a transition density $q(t, x, y)$ with respect to m , which is jointly continuous and symmetric in x and y , that is: $q(t, x, y) = q(t, y, x)$. This allows us to define, for $\lambda > 0$, the resolvent kernel of X by:

$$u_\lambda(x, y) = \int_0^\infty e^{-\lambda t} q(t, x, y) dt. \quad (1)$$

We also introduce $(L_t^a, t \geq 0)$ the local time of X at a , with the normalization:

$$L_t^a := \lim_{\varepsilon \downarrow 0} \frac{1}{m([a, a + \varepsilon])} \int_0^t 1_{[a, a + \varepsilon]}(X_s) ds$$

and $(\tau_l^{(a)}, l \geq 0)$ the right-continuous inverse of $(L_t^a, t \geq 0)$:

$$\tau_l^{(a)} := \inf\{t \geq 0; L_t^a > l\}.$$

As is well-known, $(\tau_l^{(a)}, l \geq 0)$ is a subordinator, and we denote by $\nu^{(a)}$ its Lévy measure.

To simplify the notation, we shall write in the sequel τ_l for $\tau_l^{(0)}$ and ν for $\nu^{(0)}$. We shall also denote sometimes by $\bar{\mu}(t) = \mu([t, +\infty[)$ the tail of the measure μ .

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1.2 Motivations

Our aim in this paper is to establish some penalization results involving null recurrent diffusions. Let us start by giving a definition of penalization:

Definition 1. Let $(\Gamma_t, t \geq 0)$ be a measurable process taking positive values, and such that $0 < \mathbb{E}_x[\Gamma_t] < \infty$ for any $t > 0$ and every $x \geq 0$. We say that the process $(\Gamma_t, t \geq 0)$ satisfies the penalization principle if there exists a probability measure $\mathbb{Q}_x^{(\Gamma)}$ defined on $(\Omega, \mathcal{F}_\infty)$ such that:

$$\forall s \geq 0, \forall \Lambda_s \in \mathcal{F}_s, \quad \lim_{t \rightarrow +\infty} \frac{\mathbb{E}_x[1_{\Lambda_s} \Gamma_t]}{\mathbb{E}_x[\Gamma_t]} = \mathbb{Q}_x^{(\Gamma)}(\Lambda_s).$$

This problem has been widely studied by Roynette, Vallois and Yor when \mathbb{P}_x is the Wiener measure or the law of a Bessel process (see [RVY06c] for a synthesis and further references). They showed in particular that Brownian motion may be penalized by a great number of functionals involving local times, supremums, additive functionals, numbers of downcrossings on an interval... Most of these results were then unified by Najnudel, Roynette and Yor (see [NRY09]) in a general penalization theorem, whose proof relies on the construction of a remarkable measure \mathcal{W} .

Later on, Salminen and Vallois managed in [SV09] to extend the class of diffusions for which penalization results hold. They proved in particular that under the assumption that the (restriction of the) Lévy measure $\frac{1}{\nu([1, +\infty])} \nu_{|[1, +\infty[}$ of the subordinator $(\tau_l, l \geq 0)$ is subexponential, the penalization principle holds for the functional $(\Gamma_t = h(L_t^0), t \geq 0)$ with h a non-negative and non-increasing function with compact support.

Let us recall that a probability measure μ is said to be subexponential (μ belongs to class \mathcal{S}) if, for every $t \geq 0$,

$$\lim_{t \rightarrow +\infty} \frac{\mu^{*2}([t, +\infty])}{\mu([t, +\infty])} = 2,$$

where μ^{*2} denotes the convolution of μ with itself. The main examples of subexponential distributions are given by measures having a regularly varying tail (see Chistyakov [Čis64] or Embrechts, Goldie and Veraverbek [EGV79]):

$$\mu([t, +\infty]) \underset{t \rightarrow +\infty}{\sim} \frac{\eta(t)}{t^\beta}$$

where $\beta \geq 0$ and η is a slowly varying function. When $\beta \in]0, 1[$, we shall say that such a measure belongs to class \mathcal{R} . Let us also remark that a subexponential measure always satisfies the following property:

$$\forall x \in \mathbb{R}, \quad \lim_{t \rightarrow +\infty} \frac{\mu([t+x, +\infty])}{\mu([t, +\infty])} = 1.$$

The set of such measures shall be denoted by \mathcal{L} , hence:

$$\mathcal{R} \subset \mathcal{S} \subset \mathcal{L}.$$

Now, following Salminen and Vallois, one may reasonably wonder what kind of penalization results may be obtained for diffusions whose normalized Lévy measure belongs to classes \mathcal{R} or \mathcal{L} . This is the main purpose of this paper, i.e. we shall prove that the results of Najnudel, Roynette and Yor remain true for diffusions whose normalized Lévy measure belongs to \mathcal{R} , and we shall give an “integrated version” when it belongs to \mathcal{L}^2 .

²In the remainder of the paper, we shall make a slight abuse of the notation and say that the measure ν belongs to \mathcal{L} or \mathcal{R} instead of $\frac{1}{\nu([1, +\infty])} \nu_{|[1, +\infty[}$ belongs to \mathcal{L} or \mathcal{R} . This is of no importance since the fact that a probability measure belongs to classes \mathcal{L} or \mathcal{R} only involves the behavior of its tail at $+\infty$.

1.3 Statement of the main results

Let $a \geq 0$, $g_a^{(t)} := \sup\{u \leq t; X_u = a\}$ and $(F_t, t \geq 0)$ be a positive and predictable process such that

$$0 < \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] < \infty.$$

Theorem 2.

1. If ν belongs to class \mathcal{L} , then

$$\forall a \geq 0, \quad \int_0^t \nu^{(a)}([s, +\infty[) ds \underset{t \rightarrow +\infty}{\sim} \int_0^t \nu([s, +\infty[) ds$$

and

$$\mathbb{E}_x \left[\int_0^t F_{g_a^{(s)}} ds \right] \underset{t \rightarrow +\infty}{\sim} \left(\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] \right) \int_0^t \nu([s, +\infty[) ds.$$

2. If ν belongs to class \mathcal{R} :

$$\forall a \geq 0, \quad \nu^{(a)}([t, +\infty[) \underset{t \rightarrow +\infty}{\sim} \nu([t, +\infty[)$$

and if F is decreasing:

$$\mathbb{E}_x \left[F_{g_a^{(t)}} \right] \underset{t \rightarrow +\infty}{\sim} \left(\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] \right) \nu([t, +\infty[)$$

Remark 3. Point 2. does not hold for every $\nu \in \mathcal{L}$. Indeed, otherwise, taking $a = 0$ and $F_t = 1_{\{L_t^0 \leq \ell\}}$ with $\ell > 0$, one would obtain:

$$\mathbb{P}_0(L_t^0 \leq \ell) = \mathbb{P}_0(\tau_\ell > t) \underset{t \rightarrow +\infty}{\sim} \ell \nu([t, +\infty[),$$

a relation which is known to hold if and only if $\nu \in \mathcal{S}$, see [EGV79] or [Sat99, p.164].

Remark 4. If $(X_t, t \geq 0)$ is a positively recurrent diffusion, then $\int_0^{+\infty} \nu([s, +\infty[) ds = m(\mathbb{R}^+)$ and the limit in Point 1. equals:

$$\lim_{t \rightarrow +\infty} \mathbb{E}_x \left[\int_0^t F_{g_a^{(s)}} ds \right] = \mathbb{E}_x \left[\int_0^{+\infty} F_{g_a^{(s)}} ds \right] = \mathbb{E}_x[F_0] \mathbb{E}_x[T_a] + \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] m(\mathbb{R}^+).$$

In the following penalization result, we shall choose the weighting functional Γ according to ν :

Theorem 5. Assume that:

a) either ν belongs to class \mathcal{L} , and $\Gamma_t = \int_0^t F_{g_a^{(s)}} ds$,

b) or ν belongs to class \mathcal{R} and $\Gamma_t = F_{g_a^{(t)}}$ with F decreasing.

Then, the penalization principle is satisfied by the functional $(\Gamma_t, t \geq 0)$, i.e. there exists a probability measure $\mathbb{Q}_x^{(F)}$ on $(\Omega, \mathcal{F}_\infty)$, which is the same in both cases, such that,

$$\forall s \geq 0, \forall \Lambda_s \in \mathcal{F}_s, \quad \lim_{t \rightarrow +\infty} \frac{\mathbb{E}_x[1_{\Lambda_s} \Gamma_t]}{\mathbb{E}_x[\Gamma_t]} = \mathbb{Q}_x^{(F)}(\Lambda_s).$$

Furthermore:

1. The measure $\mathbb{Q}_x^{(F)}$ is weakly absolutely continuous with respect to \mathbb{P}_x :

$$\mathbb{Q}_{x|\mathcal{F}_t}^{(F)} = \frac{M_t(F_{g_a})}{\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right]} \cdot \mathbb{P}_{x|\mathcal{F}_t}$$

where the martingale $(M_t(F_{g_a}), t \geq 0)$ is given by:

$$M_t(F_{g_a}) = F_{g_a^{(t)}}(s(X_t) - s(a))^+ + \mathbb{E}_x\left[\int_t^{+\infty} F_u dL_u^a | \mathcal{F}_t\right].$$

2. Define $g_a := \sup\{s \geq 0, X_s = a\}$. Then, under $\mathbb{Q}_x^{(F)}$:

- i) g_a is finite a.s.,
- ii) conditionally to g_a , the processes $(X_t, t \leq g_a)$ and $(X_{g_a+t}, t \geq 0)$ are independent,
- iii) the process $(X_{g_a+u}, u \geq 0)$ is transient, goes towards $+\infty$ and its law does not depend on the functional F .

We shall give in Theorem 21 a precise description of $\mathbb{Q}_x^{(F)}$ through an integral representation.

Remark 6. The main example of diffusion satisfying Theorems 2 and 5 is of course the Bessel process with dimension $\delta \in]0, 2[$ reflected at 0. Indeed, setting $\beta = 1 - \frac{\delta}{2} \in]0, 1[$, the tail of its Lévy measure at 0 equals:

$$\nu([t, +\infty[) = \frac{2^{1-\beta}}{\Gamma(\beta)} \frac{1}{t^\beta}$$

i.e. $\nu \in \mathcal{R}$.

Remark 7. Let us also mention that this kind of results no longer holds for positively recurrent diffusions. Indeed, it is shown in [Pro10] that if $(X_t, t \geq 0)$ is a recurrent diffusion reflected on an interval, then, under mild assumptions, the penalization principle is satisfied by the functional $(\Gamma_t = e^{-\alpha L_t^0}, t \geq 0)$ with $\alpha \in \mathbb{R}$, but unlike in Theorem 5, the penalized process so obtained remains a positively recurrent diffusion.

Example 8. Assume that $\nu \in \mathcal{R}$ and let h be a positive and decreasing function with compact support on \mathbb{R}^+ .

- Let us take $(F_t, t \geq 0) = (h(L_t^a), t \geq 0)$.

Then $\mathbb{E}_0\left[\int_0^{+\infty} h(L_s^a) dL_s^a\right] = \int_0^{+\infty} h(\ell) d\ell < \infty$ and, since $L_{g_a^{(t)}}^a = L_t^a$,

$$\mathbb{E}_0[h(L_t^a)] \underset{t \rightarrow +\infty}{\sim} \nu([t, +\infty[) \int_0^{+\infty} h(\ell) d\ell,$$

and the martingale $(M_t(L_{g_a}^a), t \geq 0)$ is an Azéma-Yor type martingale:

$$M_t(L_{g_a}^a) = h(L_t^a)(s(X_t) - s(a))^+ + \int_{L_t^a}^{+\infty} h(\ell) d\ell.$$

- Let us take $(F_t, t \geq 0) = (h(t), t \geq 0)$.

Then $\mathbb{E}_0\left[\int_0^{+\infty} h(u) dL_u^a\right] = \int_0^{+\infty} h(u) \mathbb{E}_0[dL_u^a] = \int_0^{+\infty} h(u) q(u, 0, a) du < \infty$ and therefore:

$$\mathbb{E}_0[h(g_a^{(t)})] \underset{t \rightarrow +\infty}{\sim} \nu([t, +\infty[) \int_0^{+\infty} h(u) q(u, 0, a) du,$$

and the martingale $(M_t(g_a), t \geq 0)$ is given by:

$$M_t(g_a) = h(g_a^{(t)})(s(X_t) - s(a))^+ + \int_0^{+\infty} h(v+t)q(v, X_t, a)dv.$$

- One may also take for instance $(F_t, t \geq 0) = (h(S_t), t \geq 0)$ where $S_t := \sup_{s \leq t} X_s$ or $(F_t, t \geq 0) = h\left(\int_0^t f(X_s)ds\right)$ where $f: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a Borel function. These were the first kind of weights studied by Roynette, Vallois and Yor, see [RVY06a] and [RVY06b].

1.4 Organization

The remainder of the paper is organized as follows:

- In Section 2, we introduce some notation and recall a few known results that we shall use in the sequel. They are mainly taken from [Sal97] and [SVY07].
- Section 3 is devoted to the proof of Theorem 2. The two Points 1. and 2. are dealt with separately: when $\nu \in \mathcal{R}$, the asymptotic is obtained via a Laplace transform and a Tauberien theorem, while in the case $\nu \in \mathcal{L}$, we shall use a basic result on integrated convolution products.
- Section 4 gives the proof of Point 1. of Theorem 5, which essentially relies on a meta-theorem, see [RVY06c].
- In Section 5, we derive a integral representation for the penalized measure $\mathbb{Q}_x^{(F)}$ which implies Point 2. of Theorem 5.
- Finally, Section 6 is devoted to prove that, with our normalizations, the process $(N_t^{(a)} := (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$ is a martingale.

2 Preliminaries

In this section, we essentially recall some known results that we shall need in the sequel.

- Let $T_a := \inf\{u \geq 0; X_u = a\}$ be the first passage time of X to level a . Its Laplace transform is given by

$$\mathbb{E}_x [e^{-\lambda T_a}] = \frac{u_\lambda(a, x)}{u_\lambda(a, a)}. \quad (2)$$

Since $(X_t, t \geq 0)$ is assumed to be null recurrent, we have for $x > a$, $\mathbb{E}_x[T_a] = +\infty$.

- We define $(\hat{X}_t, t \geq 0)$ the diffusion $(X_t, t \geq 0)$ killed at a :

$$\hat{X}_t := \begin{cases} X_t & t < T_a, \\ \partial & t \geq T_a. \end{cases}$$

where ∂ is a cemetery point. We denote by $\hat{q}(t, x, y)$ its transition density with respect to m :

$$\hat{\mathbb{P}}_x(\hat{X}_t \in dy) = \hat{q}(t, x, y)m(dy) = \mathbb{P}_x(X_t \in dy; t < T_a).$$

- We also introduce $(X_t^{\uparrow a}, t \geq 0)$ the diffusion $(\hat{X}_t, t \geq 0)$ conditioned not to touch a , following the construction in [SVY07]. For $x > a$ and F_t a positive, bounded and \mathcal{F}_t -measurable r.v.:

$$\mathbb{E}_x^{\uparrow a} [F_t] = \frac{1}{s(x) - s(a)} \mathbb{E}_x [F_t(s(X_t) - s(a))1_{\{t < T_a\}}].$$

By taking $F_t = f(X_t)$, we deduce in particular that, for $x, y > a$:

$$q^{\uparrow a}(t, x, y) = \frac{\widehat{q}(t, x, y)}{(s(x) - s(a))(s(y) - s(a))} \quad \text{and} \quad m^{\uparrow a}(dy) = (s(y) - s(a))^2 m(dy).$$

Letting x tend towards a , we obtain:

$$q^{\uparrow a}(t, a, y) = \frac{n_{y,a}(t)}{s(y) - s(a)} \quad \text{where} \quad \mathbb{P}_y(T_a \in dt) =: n_{y,a}(t)dt.$$

• We finally define $(X_u^{x,t,y}, u \leq t)$ the bridge of X of length t going from x to y . Its law may be obtained as a h -transform, for $u < t$:

$$\mathbb{E}^{x,t,y}[F_u] = \mathbb{E}_x \left[\frac{q(t-u, X_u, y)}{q(t, x, y)} F_u \right]. \quad (3)$$

With these notation, we may state the two following Propositions which are essentially due to Salminen.

Proposition 9 ([Sal97]).

1. The law of $g_a^{(t)} := \sup\{u \leq t; X_u = a\}$ is given by:

$$\mathbb{P}_x(g_a^{(t)} \in du) = \mathbb{P}_x(T_a > t)\delta_0(du) + q(u, x, a)\nu^{(a)}([t-u, +\infty[)du. \quad (4)$$

2. On the event $\{X_t > a\}$, the density of the couple $(g_a^{(t)}, X_t)$ reads :

$$\mathbb{P}_x(g_a^{(t)} \in du, X_t \in dy) = \mathbb{P}_x(T_a > t, X_t \in dy)\delta_0(du) + \frac{q(u, x, a)}{s(y) - s(a)} \mathbb{P}_a^{\uparrow a}(X_{t-u} \in dy)du \quad (y > a) \quad (5)$$

We now study the pre- and post- $g_a^{(t)}$ -process:

Proposition 10. Under \mathbb{P}_x :

i) Conditionnally to $g_a^{(t)}$, the process $(X_s, s \leq g_a^{(t)})$ and $(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)})$ are independent.

ii) Conditionnally to $g_a^{(t)} = u$,

$$(X_s, s \leq u) \stackrel{(law)}{=} (X_s^{x,u,a}, s \leq u).$$

iii) Conditionnally to $g_a^{(t)} = u$ and $X_t = y > a$,

$$(X_{u+s}, s \leq t-u) \stackrel{(law)}{=} (X_s^{\uparrow a, a, t-u, y}, s \leq t-u).$$

Proof. i) Point (i) follows from Proposition 5.5 of [Mil77] applied to the diffusion

$$X_s^{(t)} := \begin{cases} X_s & s < t \\ \partial & s \geq t \end{cases}$$

so that $\xi := \inf\{s \geq 0; X_s^{(t)} \notin \mathbb{R}^+\} = t$.

ii) Point (ii) is taken from [Sal97].

iii) As for Point (iii), still from [Sal97], conditionnally to $g_a^{(t)} = u$ and $X_t = y > a$, we have:

$$(X_{u+s}, s \leq t-u) \stackrel{(\text{law})}{=} \left(\widehat{X}_s^{a,t-u,y}, s \leq t-u \right).$$

But the bridges of \widehat{X} et X^\uparrow have the same law. Indeed, for $y, x > a$:

$$\begin{aligned} \widehat{\mathbb{P}}^{x,t,y}(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n) &= \widehat{\mathbb{E}}_x \left[\frac{\widehat{q}(t-t_n, X_{t_n}, y)}{\widehat{q}(t, x, y)} 1_{\{X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\}} \right] \quad (\text{from (3)}) \\ &= \mathbb{E}_x \left[\frac{(s(X_{t_n}) - s(a))q^{\uparrow a}(t-t_n, X_{t_n}, y)}{(s(x) - s(a))q^{\uparrow a}(t, x, y)} 1_{\{X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\}} 1_{\{t_n < T_a\}} \right] \\ &= \mathbb{E}_x^{\uparrow a} \left[\frac{q^{\uparrow a}(t-t_n, X_{t_n}, y)}{q^{\uparrow a}(t, x, y)} 1_{\{X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\}} \right] \quad (\text{by definition of } \mathbb{P}_x^{\uparrow a}) \\ &= \mathbb{P}^{\uparrow a, x, t, y}(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n). \end{aligned}$$

and the result follows by letting x tend toward a . □

3 Study of asymptotics

The aim of this section is to prove Theorem 2. We start with the case $\nu \in \mathcal{R}$.

3.1 Proof of Theorem 2 when $\nu \in \mathcal{R}$

Let $(F_t, t \geq 0)$ be a decreasing, positive and predictable process such that

$$0 < \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] < \infty.$$

Our approach in this section is based on the study of the Laplace transform of $t \mapsto \mathbb{E}_x \left[F_{g_a^{(t)}} \right]$. Indeed, from Propositions 9 and 10, we may write, applying Fubini's Theorem:

$$\begin{aligned} &\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x \left[F_{g_a^{(t)}} \right] dt \\ &= \int_0^{+\infty} e^{-\lambda t} \int_0^t \mathbb{E}_x \left[F_u | g_a^{(t)} = u \right] \mathbb{P}(g_a^{(t)} \in du) dt \\ &= \mathbb{E}_x[F_0] \int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x(T_a > t) dt + \int_0^{+\infty} e^{-\lambda t} \int_0^t \mathbb{E}_x[F_u | X_u = a] q(u, x, a) \nu^{(a)}([t-u, +\infty[) du dt \\ &= \mathbb{E}_x[F_0] \frac{1 - \mathbb{E}_x[e^{-\lambda T_a}]}{\lambda} + \int_0^{+\infty} e^{-\lambda t} \mathbb{P}^{x,t,a}(F_t) q(t, x, a) dt \times \int_0^{+\infty} e^{-\lambda t} \nu^{(a)}([t, +\infty[) dt \end{aligned} \quad (6)$$

We shall now study the asymptotic (when $\lambda \rightarrow 0$) of each term separately. To this end, we state and prove two Lemmas.

3.1.1 The Laplace transform of $t \rightarrow \nu^{(a)}([t, +\infty[)$

Lemma 11. *The following formula holds:*

$$\frac{1}{\lambda u_\lambda(a, a)} = \int_0^{+\infty} e^{-\lambda t} \nu^{(a)}([t, +\infty[) dt$$

Proof. Since τ is a subordinator and m has no atoms, from the Lévy-Khintchine formula:

$$\mathbb{E}_a \left[e^{-\lambda \tau_t^{(a)}} \right] = \exp \left(l \int_0^{+\infty} (1 - e^{-\lambda t}) \nu^{(a)}(dt) \right).$$

Then, from the classic relation:

$$\mathbb{E}_a \left[e^{-\lambda \tau_t^{(a)}} \right] = e^{-l/u_{\lambda(a, a)}}$$

we deduce that

$$\frac{1}{u_\lambda(a, a)} = \int_0^{+\infty} (1 - e^{-\lambda t}) \nu^{(a)}(dt).$$

Now, let $\varepsilon > 0$:

$$\begin{aligned} \int_\varepsilon^\infty (1 - e^{-\lambda t}) \nu^{(a)}(dt) &= [(e^{-\lambda t} - 1) \nu^{(a)}([t, +\infty[)]_\varepsilon^{+\infty} + \int_\varepsilon^\infty \lambda e^{-\lambda t} \nu^{(a)}([t, +\infty[) dt \\ &= (1 - e^{-\lambda \varepsilon}) \nu^{(a)}([\varepsilon, +\infty[) + \int_\varepsilon^\infty \lambda e^{-\lambda t} \nu^{(a)}([t, +\infty[) dt \end{aligned}$$

Since both terms are positive, we may let $\varepsilon \rightarrow 0$ to obtain:

$$\frac{1}{\lambda u_\lambda(a, a)} = \int_0^\infty e^{-\lambda t} \nu^{(a)}([t, +\infty[) dt + \ell,$$

where $\ell := \lim_{\varepsilon \rightarrow 0} \varepsilon \nu([\varepsilon, +\infty[)$, and it remains to prove that $\ell = 0$. Assume that $\ell > 0$. Then:

$\nu^{(a)}([\varepsilon, +\infty[) \underset{\varepsilon \rightarrow 0}{\sim} \frac{\ell}{\varepsilon}$ and :

$$\begin{aligned} \int_\varepsilon^1 t \nu^{(a)}(dt) &= [-t \nu^{(a)}([t, 1])]_\varepsilon^1 + \int_\varepsilon^1 \nu^{(a)}([t, 1]) dt \\ &= \varepsilon \nu^{(a)}([\varepsilon, 1]) + \int_\varepsilon^1 \nu^{(a)}([t, 1]) dt \\ &\xrightarrow{\varepsilon \rightarrow 0} +\infty, \end{aligned}$$

since, from our hypothesis, $\nu^{(a)}([t, 1]) \underset{t \rightarrow 0}{\sim} \frac{\ell}{t}$, i.e. $t \mapsto \nu^{(a)}([t, 1])$ is not integrable at 0. This contradicts the fact that $\nu^{(a)}$ is the Lévy measure of a subordinator, hence $\ell = 0$ and the proof is completed. \square

Remark 12. Since we assume that $(X_t, t \geq 0)$ is a null recurrent diffusion, we have $m(\mathbb{R}^+) = +\infty$ and from Salminen [Sal93]:

$$\lim_{\lambda \rightarrow 0} \lambda u_\lambda(a, a) = \frac{1}{m(\mathbb{R}^+)} = 0. \quad (7)$$

Thus, from the monotone convergence theorem, the function $t \rightarrow \nu^{(a)}([t, +\infty[)$ is not integrable at $+\infty$. On the other hand, if $(X_t, t \geq 0)$ is positively recurrent, we obtain:

$$\int_0^{+\infty} \nu^{(a)}([t, +\infty[) dt = m(\mathbb{R}^+) < +\infty.$$

We now study the asymptotic of the first hitting time of X to level a .

Lemma 13. *Let $x > a$ and assume that ν belongs to class \mathcal{R} . Then:*

i) *The tails of ν and $\nu^{(a)}$ are equivalent:*

$$\nu^{(a)}([t, +\infty[) \underset{t \rightarrow +\infty}{\sim} \nu([t, +\infty[).$$

ii) *The survival function of T_a satisfies the following property:*

$$\mathbb{P}_x(T_a \geq t) \underset{t \rightarrow +\infty}{\sim} (s(x) - s(a))\nu([t, +\infty[). \quad (8)$$

Proof. We shall use the following Tauberian theorem (see Feller [Fel71, Chap. XIII.5, p.446] or [BGT89, Section 1.7]):

Let f be a positive and decreasing function, $\beta \in]0, 1[$ and η a slowly varying function. Then,

$$f(t) \underset{t \rightarrow +\infty}{\sim} \frac{\eta(t)}{t^\beta} \iff \int_0^\infty e^{-\lambda t} f(t) dt \underset{\lambda \rightarrow 0}{\sim} \frac{\Gamma(\beta)}{\lambda^{1-\beta}} \eta\left(\frac{1}{\lambda}\right). \quad (9)$$

In particular, with $f(t) = \nu([t, +\infty[)$ (since $\nu \in \mathcal{R}$), we obtain:

$$\int_0^\infty e^{-\lambda t} \nu([t, +\infty[) dt = \frac{1}{\lambda u_\lambda(0, 0)} \underset{\lambda \rightarrow 0}{\sim} \frac{\Gamma(\beta)}{\lambda^{1-\beta}} \eta\left(\frac{1}{\lambda}\right).$$

Now, from Krein's Spectral Theory (see for instance [DM76, Chap.5], [KK74], [KW82] or [Kas76]), $u_\lambda(x, y)$ admits the representation, for $x \leq y$:

$$u_\lambda(x, y) = \Phi(x, \lambda) (u_\lambda(0, 0)\Phi(y, \lambda) - \Psi(y, \lambda)) \quad (10)$$

where the eigenfunctions Φ and Ψ are solutions of:

$$\begin{cases} \Phi(x, \lambda) = 1 + \lambda \int_0^x s'(dy) \int_0^y \Phi(z, \lambda) m(dz), \\ \Psi(x, \lambda) = s(x) + \lambda \int_0^x s'(dy) \int_0^y \Psi(z, \lambda) m(dz), \end{cases}$$

We deduce then, since $\lim_{\lambda \rightarrow 0} \Phi(x, \lambda) = 1$, $\lim_{\lambda \rightarrow 0} \Psi(x, \lambda) = s(x)$ and $\lim_{\lambda \rightarrow 0} u_\lambda(0, 0) = +\infty$ that:

$$\frac{u_\lambda(a, a)}{u_\lambda(0, 0)} = \Phi(a, \lambda)^2 - \frac{\Phi(a, \lambda)\Psi(a, \lambda)}{u_\lambda(0, 0)} \xrightarrow{\lambda \rightarrow 0} 1.$$

Therefore, from the Tauberien theorem (9) with $f(t) = \nu^{(a)}([t, +\infty[)$, we obtain:

$$\nu^{(a)}([t, +\infty[) \underset{t \rightarrow +\infty}{\sim} \frac{\eta(t)}{t^\beta}$$

i.e. Point (i) of Lemma 13.

To prove Point (ii), let us compute the Laplace transform of $\mathbb{P}_x(T_a \geq t)$, using (2):

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x(T_a \geq t) dt = \frac{1 - \mathbb{E}_x[e^{-\lambda T_a}]}{\lambda} = \frac{1}{\lambda} - \frac{u_\lambda(x, a)}{\lambda u_\lambda(a, a)} = \frac{u_\lambda(a, a) - u_\lambda(x, a)}{\lambda u_\lambda(a, a)}. \quad (11)$$

Now, for $x > a$, we get from (10):

$$\begin{aligned} u_\lambda(a, a) - u_\lambda(x, a) &= \Phi(a, \lambda)(u_\lambda(0, 0)\Phi(a, \lambda) - \Psi(a, \lambda)) - \Phi(a, \lambda)(u_\lambda(0, 0)\Phi(x, \lambda) - \Psi(x, \lambda)) \\ &= \Phi(a, \lambda)u_\lambda(0, 0)(\Phi(a, \lambda) - \Phi(x, \lambda)) + \Phi(a, \lambda)(\Psi(x, \lambda) - \Psi(a, \lambda)) \\ &= \Phi(a, \lambda)u_\lambda(0, 0)\left(\lambda \int_a^x s'(y) dy \int_0^y \Phi(z, \lambda) m(dz)\right) + \Phi(a, \lambda)(\Psi(x, \lambda) - \Psi(a, \lambda)), \end{aligned}$$

and, letting λ tend toward 0 and using (7):

$$\lim_{\lambda \rightarrow 0} u_\lambda(a, a) - u_\lambda(a, x) = s(x) - s(a).$$

Therefore,

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x(T_a \geq t) dt \underset{\lambda \rightarrow 0}{\sim} \frac{s(x) - s(a)}{\lambda u_\lambda(a, a)} \underset{\lambda \rightarrow 0}{\sim} (s(x) - s(a)) \frac{\Gamma(\beta)}{\lambda^{1-\beta}} \eta\left(\frac{1}{\lambda}\right)$$

and Point (ii) follows once again from the Tauberian theorem (9). □

3.1.2 Proof of Point 2. of Theorem 2

We now let λ tend toward 0 in (6). Observe first that, from our hypothesis on $(F_u, u \geq 0)$:

$$\int_0^{+\infty} \mathbb{P}^{x, u, a}(F_u) q(u, x, a) du = \int_0^{+\infty} \mathbb{E}_x[F_u | X_u = a] \mathbb{E}_x[dL_u^a] = \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] < +\infty.$$

Then, from Lemmas 11 and 13, we obtain

- if $x \leq a$,

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x[F_{g_a^{(t)}}] dt \underset{\lambda \rightarrow 0}{\sim} \frac{1}{\lambda u_\lambda(a, a)} \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right]$$

$$\text{since } \lim_{\lambda \rightarrow 0} \int_0^{+\infty} e^{-\lambda t} \mathbb{P}_x(T_a \geq t) dt = \mathbb{E}_x[T_a] < +\infty,$$

- if $x > a$,

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x[F_{g_a^{(t)}}] dt \underset{\lambda \rightarrow 0}{\sim} \frac{1}{\lambda u_\lambda(a, a)} \left(\mathbb{E}_x[F_0](s(x) - s(a)) + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] \right).$$

Therefore, for every $x \geq 0$:

$$\int_0^{+\infty} e^{-\lambda t} \mathbb{E}_x[F_{g_a^{(t)}}] dt \underset{\lambda \rightarrow 0}{\sim} \left(\mathbb{E}_x[F_0](s(x) - s(a))^+ + \mathbb{E}_x\left[\int_0^{+\infty} F_u dL_u^a\right] \right) \frac{\Gamma(\beta)}{\lambda^{1-\beta}} \eta\left(\frac{1}{\lambda}\right)$$

and Point 2. follows from the Tauberian theorem (9) since $t \mapsto \mathbb{E}_x[F_{g_a^{(t)}}]$ is decreasing. □

3.2 Proof of Theorem 2 when $\nu \in \mathcal{L}$

Let $(F_t, t \geq 0)$ be a positive and predictable process such that

$$0 < \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] < \infty.$$

From Propositions 9 and 10 we have the decomposition:

$$\begin{aligned} \int_0^t \mathbb{E}_x \left[F_{g_a^{(s)}} \right] ds &= \int_0^t \int_0^s \mathbb{E}_x \left[F_u | g_a^{(s)} = u \right] \mathbb{P}(g_a^{(s)} \in du) ds \\ &= \mathbb{E}_x [F_0] \int_0^t \mathbb{P}_x(T_a > s) ds + \int_0^t \int_0^s \mathbb{E}_x [F_u | X_u = a] q(u, a, x) \nu^{(a)}([s - u, +\infty[) du ds. \end{aligned} \quad (12)$$

But, inverting the Laplace transform (11), we deduce that:

$$\mathbb{P}_x(T_a > s) = \int_0^s (q(u, a, a) - q(u, a, x)) \nu^{(a)}([s - u, +\infty[) du,$$

hence, we may rewrite:

$$\int_0^t \mathbb{E}_x \left[F_{g_a^{(s)}} \right] ds = \int_0^t f * \overline{\nu}^{(a)}(s) ds$$

with $f(u) = \mathbb{E}_x[F_0](q(u, a, a) - q(u, a, x)) + \mathbb{P}^{x, u, a}(F_u)q(u, x, a)$ and $\overline{\nu}^{(a)}(u) = \nu^{(a)}([u, +\infty[)$. As in the previous section, the study of the asymptotic (when $t \rightarrow +\infty$) will rely on a few Lemmas.

3.2.1 Asymptotic of an integrated convolution product

Lemma 14. *Let μ be a measure whose tail $\overline{\mu}(t) = \mu([t, +\infty[)$ satisfies the following property:*

$$\text{for every } u \geq 0, \quad \int_0^{t-u} \overline{\mu}(s) ds \underset{t \rightarrow +\infty}{\sim} \int_0^t \overline{\mu}(s) ds,$$

and let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a continuous function such that $\int_0^{+\infty} f(u) du < +\infty$. Then,

$$\int_0^t f * \overline{\mu}(s) ds \underset{t \rightarrow +\infty}{\sim} \int_0^{+\infty} f(u) du \int_0^t \overline{\mu}(s) ds.$$

Proof. Let $\varepsilon > 0$. There exists $A > 0$ such that, for every $t \geq A$, $\left| \int_t^{+\infty} f(u) du \right| < \varepsilon$. From Fubini's Theorem, we may write:

$$\begin{aligned} \int_0^t f * \overline{\mu}(s) ds &= \int_0^t f(u) du \int_u^t \overline{\mu}(s - u) ds \\ &= \int_0^t f(u) du \int_0^{t-u} \overline{\mu}(s) ds \\ &= \int_0^A f(u) du \int_0^{t-u} \overline{\mu}(s) ds + \int_A^t f(u) du \int_0^{t-u} \overline{\mu}(s) ds \end{aligned}$$

Using this decomposition, we obtain

$$\begin{aligned}
& \left| \int_0^{+\infty} f(u) du - \frac{\int_0^t f * \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \right| \\
& \leq \left| \int_0^A f(u) \left(1 - \frac{\int_0^{t-u} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \right) du \right| + \left| \int_A^t f(u) \frac{\int_0^{t-u} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} du \right| + \left| \int_A^{+\infty} f(u) du \right| \\
& \leq \int_0^A |f(u)| \left(1 - \frac{\int_0^{t-A} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \right) du + \left| \int_A^t f(u) \frac{\int_0^{t-u} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} du \right| + \varepsilon.
\end{aligned} \tag{13}$$

Then, applying the second mean value theorem, there exists $c \in]A, t[$ such that

$$\int_A^t f(u) \frac{\int_0^{t-u} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} du = \frac{\int_0^{t-A} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \int_A^c f(u) du$$

hence,

$$\left| \int_A^t f(u) \frac{\int_0^{t-u} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} du \right| = \frac{\int_0^{t-A} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \left| \int_A^{+\infty} f(u) du - \int_c^{+\infty} f(u) du \right| \leq 2\varepsilon \frac{\int_0^{t-A} \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds}$$

and, letting t tend to $+\infty$ in (13), we finally obtain:

$$\lim_{t \rightarrow +\infty} \left| \int_0^{+\infty} f(u) du - \frac{\int_0^t f * \bar{\mu}(s) ds}{\int_0^t \bar{\mu}(s) ds} \right| \leq 3\varepsilon.$$

□

Remark 15. Assume that $\nu \in \mathcal{L}$. Then ν satisfies the hypothesis of Lemma 14. Indeed for $u \geq 0$, since $\bar{\nu}(s-u) \underset{s \rightarrow +\infty}{\sim} \bar{\nu}(s)$ and $\bar{\nu}$ is not integrable at $+\infty$, we have:

$$\int_0^t \bar{\nu}(s) ds \underset{t \rightarrow +\infty}{\sim} \int_u^t \bar{\nu}(s) ds \underset{t \rightarrow +\infty}{\sim} \int_u^t \bar{\nu}(s-u) ds = \int_0^{t-u} \bar{\nu}(s) ds.$$

Lemma 16. The following formula holds, for $x > a$:

$$\int_0^{+\infty} (q(u, a, a) - q(u, a, x)) du = s(x) - s(a).$$

Proof. We set $f(t) = \int_0^t (q(u, a, a) - q(u, a, x)) du$. From Borodin-Salminen [BS02, p.21], we have:

$$f(t) = \mathbb{E}_a [L_t^a] - \mathbb{E}_a [L_t^x].$$

Since $(N_t^{(a)} = (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$ is a martingale (see Section 6), this relation may be rewritten:

$$\begin{aligned}
f(t) &= \mathbb{E}_a [(s(X_t) - s(a))^+] - \mathbb{E}_a [(s(X_t) - s(x))^+] \\
&= (s(x) - s(a)) \mathbb{P}_a(X_t \geq x) + \mathbb{E}_a [(s(X_t) - s(a)) 1_{\{a \leq X_t \leq x\}}].
\end{aligned}$$

Then

$$\begin{aligned}
|f(t) - (s(x) - s(a))| &\leq (s(x) - s(a))\mathbb{P}_a(X_t \leq x) + \mathbb{E}_a[(s(X_t) - s(a))1_{\{a \leq X_t \leq x\}}] \\
&\leq (s(x) - s(a))(\mathbb{P}_a(X_t \leq x) + \mathbb{P}_a(a \leq X_t \leq x)) \\
&\leq 2(s(x) - s(a))\mathbb{P}_a(X_t \leq x) \\
&\leq 2(s(x) - s(a))\mathbb{P}_0(X_t \leq x) \xrightarrow[t \rightarrow +\infty]{} 0
\end{aligned}$$

from [PRY10, Chap.8, p.226], since $(X_t, t \geq 0)$ is null recurrent. \square

Lemma 17. *Assume that ν belongs to class \mathcal{L} . Then:*

$$\forall a \geq 0, \quad \int_0^t \nu^{(a)}([s, +\infty[)ds \underset{t \rightarrow +\infty}{\sim} \int_0^t \nu([s, +\infty[)ds$$

Proof. Let us define the function:

$$f(t) = \int_0^t q(u, 0, 0) \nu^{(a)}([t - u, +\infty[)du.$$

We claim that $\lim_{t \rightarrow +\infty} f(t) = 1$. Indeed, let us decompose f as follows, with $\varepsilon > 0$:

$$\begin{aligned}
f_a(t) &= \int_0^t (q(u, 0, 0) - q(u, 0, a)) \nu^{(a)}([t - u, +\infty[)du + \mathbb{P}_0(T_a \leq t) \\
&= \int_0^{t-\varepsilon} (q(u, 0, 0) - q(u, 0, a)) \nu^{(a)}([t - u, +\infty[)du \\
&\quad + \int_{t-\varepsilon}^t (q(u, 0, 0) - q(u, 0, a)) \nu^{(a)}([t - u, +\infty[)du + \mathbb{P}_0(T_a \leq t). \\
&= \int_0^{+\infty} (q(u, 0, 0) - q(u, 0, a)) 1_{\{u \leq t-\varepsilon\}} \nu^{(a)}([t - u, +\infty[)du \\
&\quad + \int_0^\varepsilon (q(t - u, 0, 0) - q(t - u, 0, a)) \nu^{(a)}([u, +\infty[)du + \mathbb{P}_0(T_a \leq t).
\end{aligned}$$

From [PRY10, Chap.8, p.224], we know that for every $u \geq 0$ the function $z \mapsto q(u, 0, z)$ is decreasing, hence the function

$$u \mapsto q(u, 0, 0) - q(u, 0, a)$$

is a positive and integrable function from Lemma 16. Therefore, from the dominated convergence theorem, the first integral tends toward 0 as $t \rightarrow +\infty$. Moreover, it is known from Salminen [Sal96] that for every $x, y \geq 0$,

$$\lim_{t \rightarrow +\infty} q(t, x, y) = \frac{1}{m(\mathbb{R}^+)} = 0,$$

which proves, still from the dominated convergence theorem, that the second integral also tends toward 0 as $t \rightarrow +\infty$. Finally, we deduce that $\lim_{t \rightarrow +\infty} f_a(t) = \mathbb{P}_0(T_a < +\infty) = 1$.

Observe now that, since $\bar{\nu} * q(t) = \int_0^t \nu([u, +\infty[)q(t - u, 0, 0)du = 1$, we have from Fubini-Tonelli:

$$\int_0^t \nu^{(a)}([s, +\infty[)ds = 1 * \bar{\nu}^{(a)}(t) = (\bar{\nu} * q) * \bar{\nu}^{(a)}(t) = \bar{\nu} * f_a(t) = \int_0^t f_a(s) \nu([t - s, +\infty[)ds.$$

Let $\varepsilon > 0$. There exists $A > 0$ such that, for every $s \geq A$:

$$1 - \varepsilon \leq f(s) \leq 1 + \varepsilon.$$

Integrating this relation, we deduce that, for $t > A$:

$$(1 - \varepsilon) \int_A^t \bar{\nu}(t-s) ds \leq \int_A^t f_a(s) \bar{\nu}(t-s) ds \leq (1 + \varepsilon) \int_A^t \bar{\nu}(t-s) ds.$$

Therefore:

$$\left| \int_0^t f_a(s) \bar{\nu}(t-s) ds - \int_A^t \bar{\nu}(t-s) ds - \int_0^A f_a(s) \bar{\nu}(t-s) ds \right| \leq \varepsilon \int_A^t \bar{\nu}(t-s) ds = \varepsilon \int_0^{t-A} \bar{\nu}(s) ds,$$

and it only remains to divide both terms by $\int_0^t \bar{\nu}(s) ds$ and let t tend toward $+\infty$ to conclude, thanks to Remark 15, that:

$$\left| \lim_{t \rightarrow +\infty} \frac{\int_0^t \bar{\nu}^{(a)}(s) ds}{\int_0^t \bar{\nu}(s) ds} - 1 \right| \leq \varepsilon.$$

□

3.2.2 Proof of Point 1. of Theorem 2

Going back to (12), we have, with $f(u) = \mathbb{P}^{x,u,a}(F_u)q(u, x, a)$ and $\bar{\nu}^{(a)}(u) = \nu^{(a)}([u, +\infty[)$:

$$\int_0^t \mathbb{E}_x [F_{g_a(s)}] ds = \left(\mathbb{E}_x [F_0] \int_0^t \mathbb{P}_x(T_a > s) ds + \int_0^t f * \bar{\nu}^{(a)}(s) ds \right).$$

From Lemmas 14 and 16, we deduce that:

$$\lim_{t \rightarrow +\infty} \frac{1}{\int_0^t \bar{\nu}(s) ds} \int_0^t \mathbb{P}_x(T_a > s) ds = (s(x) - s(a))^+$$

since, for $x \leq a$, $\int_0^{+\infty} \mathbb{P}_x(T_a > s) ds = \mathbb{E}_x [T_a] < +\infty$. Then, Point 1. of Theorem 2 follows from Lemmas 14 and 17 and the fact that:

$$\int_0^{+\infty} f(u) du = \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u)q(u, x, a) du = \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] < +\infty.$$

□

4 The penalization principle

4.1 Preliminaries: a meta-theorem and some notations

To prove Theorem 5, we shall apply a meta-theorem, whose proof relies mainly on Scheffé's Lemma (see Meyer [Mey66, p.37]):

Theorem 18 ([RVY06c]). *Let $(\Gamma_t, t \geq 0)$ be a positive stochastic process satisfying for every $t > 0$, $0 < \mathbb{E}[\Gamma_t] < +\infty$. Assume that, for every $s \geq 0$:*

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{E}[\Gamma_t | \mathcal{F}_s]}{\mathbb{E}[\Gamma_t]} =: M_s$$

exists a.s., and that,

$$\mathbb{E}[M_s] = 1.$$

Then,

i) for every $s \geq 0$ and $\Lambda_s \in \mathcal{F}_s$:

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{E}[1_{\Lambda_s} \Gamma_t]}{\mathbb{E}[\Gamma_t]} = \mathbb{E}[M_s 1_{\Lambda_s}].$$

ii) there exists a probability measure \mathbb{Q} on $(\Omega, \mathcal{F}_\infty)$ such that for every $s \geq 0$:

$$\mathbb{Q}(\Lambda_s) = \mathbb{E}[M_s 1_{\Lambda_s}].$$

In the following, we shall use Biane-Yor's notations [BY87]. We denote by Ω_{loc} the set of continuous functions ω taking values in \mathbb{R}^+ and defined on an interval $[0, \xi(\omega)] \subset [0, +\infty]$. Let \mathbb{P} and \mathbb{Q} be two probability measures, such that $\mathbb{P}(\xi = +\infty) = 0$. We denote by $\mathbb{P} \circ \mathbb{Q}$ the image measure $\mathbb{P} \otimes \mathbb{Q}$ by the concatenation application :

$$\begin{aligned} \circ : \Omega_{\text{loc}} \times \Omega_{\text{loc}} &\longrightarrow \Omega_{\text{loc}} \\ (\omega_1, \omega_2) &\longmapsto \omega_1 \circ \omega_2 \end{aligned}$$

defined by $\xi(\omega_1 \circ \omega_2) = \xi(\omega_1) + \xi(\omega_2)$, and

$$(\omega_1 \circ \omega_2)(t) = \begin{cases} \omega_1(t) & \text{si } 0 \leq t \leq \xi(\omega_1) \\ \omega_1(\xi(\omega_1)) + \omega_2(t - \xi(\omega_1)) - \omega_2(0) & \text{si } \xi(\omega_1) \leq t \leq \xi(\omega_1) + \xi(\omega_2). \end{cases}$$

To simplify the notations, we define the following measure, which was first introduced by Najnudel, Roynette and Yor [NRY09]:

Definition 19. Let \mathcal{W}_x be the measure defined by:

$$\mathcal{W}_x = \int_0^{+\infty} du q(u, x, a) \mathbb{P}^{x, u, a} \circ \mathbb{P}_a^{\uparrow a} + (s(x) - s(a))^+ \mathbb{P}_x^{\uparrow a}$$

\mathcal{W}_x is a sigma-finite measure with infinite mass.

This measure enjoys many remarkable properties, and was the main ingredient in the proof of the penalization results they obtained for Brownian motion. A similar construction was made by Yano, Yano and Yor for symmetric stable Lévy processes, see [YYY09].

With this new notation, we shall now write:

$$\begin{aligned} \mathcal{W}_x(F_{g_a}) &= \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] + \mathbb{E}_x^{\uparrow a}[F_0](s(x) - s(a))^+ \\ &= \mathbb{E}_x \left[\int_0^{+\infty} F_u dL_u^a \right] + \mathbb{E}_x[F_0](s(x) - s(a))^+. \end{aligned}$$

4.2 Proof of Point i) of Theorem 5

Let $0 \leq u \leq t$. Using Biane-Yor's notation, we write:

$$(X_s, s \leq t) = (X_s, s \leq u) \circ (X_{s+u}, 0 \leq s \leq t - u)$$

hence, from the Markov property, denoting $F_{g_a^{(t)}} = F(X_s, s \leq t)$:

$$\mathbb{E}_x[F(X_s, s \leq t) 1_{\{u \leq t\}} | \mathcal{F}_u] = \widehat{\mathbb{E}}_{X_u} \left[F((X_s, s \leq u) \circ (\widehat{X}_s, 0 \leq s \leq t - u)) 1_{\{u \leq t\}} \right].$$

Let us assume first that $\nu \in \mathcal{R}$ and that $(F_t, t \geq 0)$ is decreasing. Then, from Theorem 2 with $\Gamma_t = F_{g_a^{(t)}}$:

$$\begin{aligned}
& \lim_{t \rightarrow +\infty} \frac{\widehat{\mathbb{E}}_{X_u} \left[F((X_s, s \leq u) \circ (\widehat{X}_s, 0 \leq s \leq t-u)) 1_{\{u \leq t\}} \right]}{\nu([t, +\infty[)} \\
&= \widehat{\mathbb{E}}_{X_u} \left[F((X_s, s \leq u) \circ \widehat{X}_0) \right] (s(X_u) - s(a))^+ + \widehat{\mathbb{E}}_{X_u} \left[\int_u^{+\infty} F((X_s, s \leq u) \circ (\widehat{X}_s, 0 \leq s \leq v-u)) d\widehat{L}_v^a \right] \\
&= F((X_s, s \leq u)(s(X_u) - s(a))^+ + \mathbb{E}_x \left[\int_u^{+\infty} F((X_s, s \leq u) \circ (X_s, 0 \leq s \leq v-u)) dL_v^a | \mathcal{F}_u \right] \\
&= F_{g_a^{(u)}}(s(X_u) - s(a))^+ + \mathbb{E}_x \left[\int_u^{+\infty} F_{g_a^{(v)}} dL_v^a | \mathcal{F}_u \right] \\
&= F_{g_a^{(u)}}(s(X_u) - s(a))^+ + \mathbb{E}_x \left[\int_u^{+\infty} F_v dL_v^a | \mathcal{F}_u \right],
\end{aligned}$$

hence,

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{E}_x \left[F_{g_a^{(t)}} | \mathcal{F}_u \right]}{\mathbb{E}_x \left[F_{g_a^{(t)}} \right]} = \frac{M_u(F_{g_a})}{\mathcal{W}_x(F_{g_a})}.$$

On the other hand, if $\nu \in \mathcal{L}$ and $\Gamma_t = \int_0^t F_{g_a^{(s)}} ds$, a similar computation gives:

$$\begin{aligned}
& \lim_{t \rightarrow +\infty} \frac{\int_0^t \widehat{\mathbb{E}}_{X_u} \left[F((X_s, s \leq u) \circ (\widehat{X}_s, 0 \leq s \leq v-u)) 1_{\{u \leq t\}} \right] dv}{\int_0^t \nu([s, +\infty[) ds} \\
&= F_{g_a^{(u)}}(s(X_u) - s(a))^+ + \mathbb{E}_x \left[\int_u^{+\infty} F_v dL_v^a | \mathcal{F}_u \right],
\end{aligned}$$

and

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{E}_x \left[\int_0^t F_{g_a^{(s)}} ds | \mathcal{F}_u \right]}{\mathbb{E}_x \left[\int_0^t F_{g_a^{(s)}} ds \right]} = \frac{M_u(F_{g_a})}{\mathcal{W}_x(F_{g_a})}.$$

Therefore, to apply Theorem 18, it remains to prove that:

$$\forall t \geq 0, \quad \mathbb{E}_x [M_t(F_{g_a})] = \mathcal{W}_x(F_{g_a}).$$

We shall make a direct computation, applying Proposition 9:

- if $x > a$,

$$\begin{aligned}
\mathbb{E}_x [M_t(F_{g_a})] &= \mathbb{E}_x \left[F_{g_a^{(t)}} (s(X_t) - s(a))^+ + \mathbb{E}_x \left[\int_t^{+\infty} F_u dL_u^a | \mathcal{F}_t \right] \right] \\
&= \int_a^{+\infty} \mathbb{E}_x [F_0 | X_t = y, T_a > t] (s(y) - s(a)) \mathbb{P}_x(T_a > t, X_t \in dy) \\
&\quad + \int_0^t \int_a^{+\infty} \mathbb{P}^{x,u,a}(F_u) q(u, a, x) \mathbb{P}_a^\uparrow(X_{t-u} \in dy) du + \int_t^{+\infty} \mathbb{P}^{x,u,a}(F_u) q(u, a, x) du \\
&= \mathbb{E}_x [F_0 (s(X_t) - s(a)) 1_{\{t < T_a\}}] + \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u) q(u, a, x) du \\
&= \mathbb{E}_x^\uparrow [F_0] (s(x) - s(a)) + \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u) q(u, a, x) du = \mathcal{W}_x(F_{g_a}),
\end{aligned}$$

• if $x \leq a$, then, for $y > a$, $\mathbb{P}_x(T_a > t, X_t \in dy) = 0$ since X has continuous paths, and the same computation leads to:

$$\mathbb{E}_x [M_t(F_{g_a})] = \int_0^{+\infty} \mathbb{P}^{x,u,a}(F_u) q(u, a, x) du = \mathcal{W}_x(F_{g_a}).$$

Therefore, for every $x \geq 0$, $\mathbb{E}_x \left[\frac{M_t(F_{g_a})}{\mathcal{W}_x(F_{g_a})} \right] = 1$, and the proof is completed. \square

Remark 20. Consider the martingale $(N_t^{(a)} = (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$. We apply the balayage formula to the semimartingale $((s(X_t) - s(a))^+, t \geq 0)$:

$$\begin{aligned}
F_{g_a^{(t)}} (s(X_t) - s(a))^+ &= F_0 (s(x) - s(a))^+ + \int_0^t F_{g_a^{(u)}} d(s(X_u) - s(a))^+ \\
&= F_0 (s(x) - s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \int_0^t F_{g_a^{(u)}} dL_u^a \\
&= F_0 (s(x) - s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \int_0^t F_u dL_u^a.
\end{aligned}$$

Therefore, the martingale $(M_t(F_{g_a}), t \geq 0)$ may be rewritten:

$$M_t(F_{g_a}) = F_0 (s(x) - s(a))^+ + \int_0^t F_{g_a^{(u)}} dN_u^{(a)} + \mathbb{E}_x \left[\int_0^{+\infty} F_s dL_s^a | \mathcal{F}_u \right].$$

5 An integral representation of $\mathbb{Q}_x^{(F)}$

Finally, Point 2. of Theorem 5 is a direct consequence of the following result:

Theorem 21. $\mathbb{Q}_x^{(F)}$ admits the following integral representation:

$$\mathbb{Q}_x^{(F)} = \frac{1}{\mathcal{W}_x(F_{g_a})} \left(\int_0^{+\infty} q(u, x, a) F_u \mathbb{P}^{x,u,a} \circ \mathbb{P}_a^{\uparrow a} + (s(x) - s(a)) F_0 \mathbb{P}_x^{\uparrow a} \right)$$

Proof. Let G, H and φ be three Borel bounded functionals. We write:

$$\begin{aligned}
& \mathcal{W}_x(F_{g_a}) \mathbb{Q}_x^{(F)} \left(G(X_s, s \leq g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)}) \right) \\
&= \mathbb{E}_x \left[G(X_s, s \leq g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)}) M_t(F_{g_a}) \right] \\
&= \mathbb{E}_x \left[G(X_s, s \leq g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)}) \left(F_{g_a^{(t)}}(s(X_t) - s(a))^+ + \mathbb{E}_x \left[\int_t^{+\infty} F_u dL_u^a | \mathcal{F}_t \right] \right) \right] \\
&= I_1(t) + I_2(t).
\end{aligned}$$

On the one hand, I_2 equals

$$I_2(t) = \mathbb{E}_x \left[G(X_s, s \leq g_a^{(t)}) \varphi(g_a^{(t)}) H(X_{g_a^{(t)}+s}, s \leq t - g_a^{(t)}) \int_t^{+\infty} F_u dL_u^a \right] \xrightarrow{t \rightarrow +\infty} 0$$

from the dominated convergence theorem.

On the other hand, from Propositions 9 and 10:

$$\begin{aligned}
I_1(t) &= \int_a^{+\infty} \int_0^t \mathbb{P}_x \left(g_a^{(t)} \in du, X_t \in dy \right) \times \\
&\quad \mathbb{E}_x \left[G(X_s, s \leq u) \varphi(u) H(X_{u+s}, s \leq t - u) F_u(s(y) - s(a)) | g_a^{(t)} = u, X_t = y \right] \\
&= \int_a^{+\infty} \int_0^t \mathbb{P}_x \left(g_a^{(t)} \in du, X_t \in dy \right) \times \\
&\quad \mathbb{P}^{x,u,a} (G(X_s, s \leq u) F_u) \varphi(u) (s(y) - s(a)) \mathbb{E}_x \left[H(X_{u+s}, s \leq t - u) | g_a^{(t)} = u, X_t = y \right].
\end{aligned}$$

We now separate the two cases $g_a^{(t)} = 0$ and $g_a^{(t)} > 0$ as in relation (5).

- First, when $g_a^{(t)} = 0$ and $x \leq a$, this term is null. Indeed, for $x \leq a < y$, $\mathbb{P}_x (T_a > t, X_t \in dy) = 0$ since X has continuous paths. Next, for $x > a$:

$$\begin{aligned}
& \int_a^{+\infty} \mathbb{P}_x (T_a > t, X_t \in dy) G(x) \mathbb{E}_x [F_0] \varphi(0) (s(y) - s(a)) \mathbb{E}_x [H(X_s, s \leq t) | T_a > t, X_t = y] \\
&= G(x) \mathbb{E}_x [F_0] \varphi(0) \mathbb{E}_x [(s(X_t) - s(a))^+ H(X_s, s \leq t) 1_{\{T_a > t\}}] \\
&= G(x) \mathbb{E}_x [F_0] \varphi(0) (s(x) - s(a)) \mathbb{E}_x^{\uparrow a} [H(X_s, s \leq t)] \\
&\xrightarrow{t \rightarrow +\infty} G(x) \mathbb{E}_x [F_0] \varphi(0) (s(x) - s(a))^+ \mathbb{E}_x^{\uparrow a} [H(X_s, s \geq 0)].
\end{aligned}$$

- Second, when $g_a^{(t)} > 0$:

$$\begin{aligned}
& \int_a^{+\infty} \int_0^t \frac{q(u, x, a)}{s(y) - s(a)} \mathbb{P}_a^{\uparrow} (X_{t-u} \in dy) du \times \\
&\quad \mathbb{P}^{x,u,a} (G(X_s, s \leq u) F_u) \varphi(u) (s(y) - s(a)) \mathbb{E}_x [H(X_{u+s}, s \leq t - u) | g_a^{(t)} = u, X_t = y] \\
&= \int_a^{+\infty} \int_0^t q(u, x, a) \mathbb{P}_a^{\uparrow} (X_{t-u} \in dy) du \times \\
&\quad \mathbb{P}^{x,u,a} (G(X_s, s \leq u) F_u) \varphi(u) \mathbb{E}_a^{\uparrow a} [H(X_s, s \leq t - u) | X_{t-u} = y] \\
&= \int_0^t du q(u, x, a) \mathbb{P}^{x,u,a} (G(X_s, s \leq u) F_u) \varphi(u) \mathbb{E}_a^{\uparrow a} [H(X_s, s \leq t - u)] \\
&\xrightarrow{t \rightarrow +\infty} \int_0^{+\infty} du q(u, x, a) \mathbb{P}^{x,u,a} (G(X_s, s \leq u) F_u) \varphi(u) \mathbb{E}_a^{\uparrow a} [H(X_s, s \geq 0)].
\end{aligned}$$

□

Remark 22. From Theorem 21, $\mathbb{Q}_x^{(F)}(g_a < +\infty) = 1$ and we deduce that, conditionally to g_a ,

1. on the event $g_a > 0$, the law of the process $(X_{g_a+u}, u \geq 0)$ under $\mathbb{Q}_x^{(F)}$ is the same as the law of $(X_u, u \geq 0)$ under $\mathbb{P}_a^{\uparrow a}$,
2. on the event $g_a = 0$, the law of the process $(X_u, u \geq 0)$ under $\mathbb{Q}_x^{(F)}$ is the same as the law of $(X_u, u \geq 0)$ under $\mathbb{P}_x^{\uparrow a}$.

Observe that the process $(F_u, u \geq 0)$ plays no role in these results.

Example 23. Let h be a positive and decreasing function on \mathbb{R}^+ .

- Let us take $(F_t, t \geq 0) = (h(L_t^a), t \geq 0)$ and assume that $\int_0^{+\infty} h(\ell) d\ell = 1$:

$$\mathbb{Q}_0^{(h(L_{g_a}^a))} = \int_0^{+\infty} du q(u, 0, a) h(L_u^a) \mathbb{P}^{0,u,a} \circ \mathbb{P}_a^{\uparrow}.$$

Thus, under $\mathbb{Q}_0^{(h(L_{g_a}^a))}$, the r.v. L_∞^a is a.s. finite and admits $\ell \mapsto h(\ell)$ as its density function. Furthermore, conditionally to $L_\infty^a = \ell$ the process $(X_t, t \leq g_a)$ has the same law as $(X_t, t \leq \tau_\ell^{(a)})$ under \mathbb{P}_0 .

- Let us take $(F_t, t \geq 0) = (h(t), t \geq 0)$ and assume that $\int_0^{+\infty} h(u) q(u, 0, a) du = 1$:

$$\mathbb{Q}_0^{(h(g_a))} = \int_0^{+\infty} du q(u, 0, a) h(u) \mathbb{P}^{0,u,a} \circ \mathbb{P}_a^{\uparrow}.$$

Then, under $\mathbb{P}_0^{(h(g_a))}$, the r.v. g_a admits as density function $u \mapsto h(u) q(u, 0, a)$ and, conditionally to $g_a = u$ the process $(X_t, t \leq g_a)$ has the same law as $(X_t, t \leq u)$ under $\mathbb{P}^{0,u,a}$.

6 Appendix

Let $a \geq 0$ and define $(N_t^{(a)} := (s(X_t) - s(a))^+ - L_t^a, t \geq 0)$. The aim of this section is to prove that $(N_t^{(a)}, t \geq 0)$ is a martingale in the filtration $(\mathcal{F}_t, t \geq 0)$. Applying the Markov property to the diffusion $(X_t, t \geq 0)$ we deduce that:

$$\mathbb{E}_0 [N_{t+s}^{(a)} | \mathcal{F}_s] = \mathbb{E}_{X_s} [(s(X_t) - s(a))^+] - L_s^a - \mathbb{E}_{X_s} [L_t^a].$$

We set $x = X_s$, so we need to prove that for every $x \geq 0$:

$$(s(x) - s(a))^+ = \mathbb{E}_x [(s(X_t) - s(a))^+] - \mathbb{E}_x [L_t^a],$$

or rather:

$$\int_0^{+\infty} (s(y) - s(a))^+ q(t, x, y) m(dy) = \int_0^t q(u, x, a) du + (s(x) - s(a))^+.$$

Let us take the Laplace transform of this last relation (applying Fubini-Tonelli):

$$\int_0^{+\infty} (s(y) - s(a))^+ u_\lambda(x, y) m(dy) = \frac{u_\lambda(x, a)}{\lambda} + \frac{(s(x) - s(a))^+}{\lambda}. \quad (14)$$

Our aim now is to prove (14). To this end, we shall use the following representation of the resolvent kernel $u_\lambda(x, y)$ (see [BS02, p.19]):

$$u_\lambda(x, y) = \omega_\lambda^{-1} \psi_\lambda(x) \varphi_\lambda(y) \quad x \leq y$$

where ψ_λ and φ_λ are the fundamental solutions of the generalized differential equation

$$\frac{d^2}{dm \, ds} u = \lambda u \quad (15)$$

such that ψ_λ is increasing (resp. φ_λ is decreasing) and the Wronskian ω_λ is given, for all $z \geq 0$ by:

$$\omega_\lambda = \varphi_\lambda(z) \frac{d\psi_\lambda}{ds}(z) - \psi_\lambda(z) \frac{d\varphi_\lambda}{ds}(z).$$

Note that since m has no atoms, the meaning of (15) is as follows:

$$\forall y \geq x, \quad \lambda \int_x^y u(z) m(dz) = \frac{du}{ds}(y) - \frac{du}{ds}(x) \quad \text{where} \quad \frac{du}{ds}(x) := \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{s(x+h) - s(x)}.$$

- Assume first that $x \leq a$.

$$\begin{aligned} & \int_a^{+\infty} (s(y) - s(a)) u_\lambda(x, y) m(dy) \\ &= \frac{1}{\omega_\lambda} \int_a^{+\infty} \left(\int_a^y ds(z) \right) \psi_\lambda(x) \varphi_\lambda(y) m(dy) \\ &= \frac{\psi_\lambda(x)}{\omega_\lambda} \int_a^{+\infty} ds(z) \int_z^{+\infty} \varphi_\lambda(y) m(dy) \quad (\text{applying Fubini-Tonelli's theorem since } \varphi_\lambda \geq 0) \\ &= -\frac{\psi_\lambda(x)}{\lambda \omega_\lambda} \int_a^{+\infty} ds(z) \frac{d\varphi_\lambda}{ds}(z) \quad \left(\text{since } \lim_{y \rightarrow +\infty} \frac{d\varphi_\lambda}{ds}(y) = 0 \text{ as } +\infty \text{ is a natural boundary} \right) \\ &= \frac{\psi_\lambda(x)}{\lambda \omega_\lambda} \varphi_\lambda(a) \quad \left(\text{since } \lim_{z \rightarrow +\infty} \varphi_\lambda(z) = 0 \text{ as } +\infty \text{ is a natural boundary} \right) \\ &= \frac{u_\lambda(x, a)}{\lambda} \end{aligned}$$

which gives (14) for $x \leq a$.

- Now, let us suppose that $x > a$. We have, with the same computation:

$$\begin{aligned} & \int_a^{+\infty} (s(y) - s(a)) u_\lambda(x, y) m(dy) \\ &= \int_a^x (s(y) - s(a)) u_\lambda(x, y) m(dy) + \int_x^{+\infty} (s(y) - s(a)) u_\lambda(x, y) m(dy) \\ &= I_1 + I_2. \end{aligned}$$

On the one hand:

$$\begin{aligned}
I_1 &= \frac{\varphi_\lambda(x)}{\omega_\lambda} \int_a^x ds(z) \int_z^x \psi_\lambda(y) m(dy) \\
&= \frac{\varphi_\lambda(x)}{\lambda \omega_\lambda} \int_a^x ds(z) \left(\frac{d\psi_\lambda}{ds}(x) - \frac{d\psi_\lambda}{ds}(z) \right) \\
&= \frac{\varphi_\lambda(x)}{\lambda \omega_\lambda} \left((s(x) - s(a)) \frac{d\psi_\lambda}{ds}(x) - (\psi_\lambda(x) - \psi_\lambda(a)) \right) \\
&= \frac{s(x) - s(a)}{\lambda \omega_\lambda} \varphi_\lambda(x) \frac{d\psi_\lambda}{ds}(x) - \frac{u_\lambda(x, x)}{\lambda} + \frac{u_\lambda(x, a)}{\lambda}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
I_2 &= \int_x^{+\infty} (s(y) - s(x)) u_\lambda(x, y) m(dy) + (s(x) - s(a)) \int_x^{+\infty} u_\lambda(x, y) m(dy) \\
&= \frac{u_\lambda(x, x)}{\lambda} + \frac{s(x) - s(a)}{\omega_\lambda} \psi_\lambda(x) \int_x^{+\infty} \varphi_\lambda(y) m(dy) \quad (\text{from the previous computations}) \\
&= \frac{u_\lambda(x, x)}{\lambda} - \frac{s(x) - s(a)}{\lambda \omega_\lambda} \psi_\lambda(x) \frac{d\varphi_\lambda}{ds}(x).
\end{aligned}$$

Finally, gathering both terms, we obtain for $x > a$:

$$\begin{aligned}
\int_a^{+\infty} (s(y) - s(a)) u_\lambda(x, y) m(dy) &= \frac{s(x) - s(a)}{\lambda \omega_\lambda} \left(\varphi_\lambda(x) \frac{d\psi_\lambda}{ds}(x) - \psi_\lambda(x) \frac{d\varphi_\lambda}{ds}(x) \right) + \frac{u_\lambda(x, a)}{\lambda}, \\
&= \frac{s(x) - s(a)}{\lambda} + \frac{u_\lambda(x, a)}{\lambda},
\end{aligned}$$

which is the desired result (14) from the definition of the Wronskian. □

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